Quantum Gates and Clifford Algebras.

Alexander Yu. Vlasov
E-mail: `Alexander.Vlasov@pobox.spbu.ru`

12–23 July, 1999

Abstract

Clifford algebras are used for definition of spinors. Because of using spin-1/2 systems as an adequate model of quantum bit, a relation of the algebras with quantum information science has physical reasons. But there are simple mathematical properties of the algebras those also justifies such applications.

First, any complex Clifford algebra with $2n$ generators, $\mathbb{C}\mathrm{l}(2n, \mathbb{C})$, has representation as algebra of all $2^n \times 2^n$ complex matrices and so includes unitary matrix of any quantum n-gate. An arbitrary element of whole algebra corresponds to general form of linear complex transformation. The last property is also useful because linear operators are not necessary should be unitary if they used for description of restriction of some unitary operator to subspace.

The second advantage is simple algebraic structure of $\mathbb{C}\mathrm{l}(2n)$ that can be expressed via tenzor product of standard "building units" and similar with behavior of composite quantum systems. The compact notation with $2n$ generators also can be used in software for modeling of simple quantum circuits by modern conventional computers.

The standard blocks may be based on three classical groups: 2×2 complex and real unimodular matrices and group of Weyl spinors, $SU(2)$. The last group may have more close relation with nonrelativistic quantum systems as spinor representation of group of 3D rotations, $SO(3)$. The second one, $SL(2, \mathbb{R})$, is widely used for theory of quantum error correction codes [1] together with a complexification, the first group $SL(2, \mathbb{C})$ of Pauli spinors.

1 Clifford algebras.

1.1 Preliminaries.

Generally, Clifford algebra is defined [2] for linear vector space V with arbitrary quadratic form $Q(\boldsymbol{x})$ as some algebra A with map $\alpha : V \to A$, $\alpha(\boldsymbol{x})^2 = -Q(\boldsymbol{x})\mathbf{1}$ (here $\mathbf{1}$ is *unit* of the algebra), but let us consider first Euclidean case with $Q(\boldsymbol{x}) = x_1^2 + \cdots + x_n^2$. The Clifford algebra, $\mathbb{Cl}(n)$, is generated by n elements $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ with property:

$$\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i = -2\delta_{ij}; \implies \boldsymbol{e}_i\boldsymbol{e}_j = -\boldsymbol{e}_j\boldsymbol{e}_i \ (i \neq j), \quad \boldsymbol{e}_k^2 = -\mathbf{1} \tag{1}$$

The eq.(1) defines 2^n-dimensional real algebra. It is clear, because product of any number of $\boldsymbol{e}_i$ can be simplified to product[1] with up to n different $\boldsymbol{e}_i$:

$$\boldsymbol{e}_{i_1}\boldsymbol{e}_{i_2}\cdots\boldsymbol{e}_{i_k}; \quad 0 \leq k \leq n; \quad i_1 < i_2 < \cdots < i_k \tag{2}$$

There are 2^n such terms because every combination corresponds to n-digits binary number with units in positions $i_1, i_2, \ldots, i_k$:

$$\underbrace{\underbrace{\underbrace{00\ldots01}_{i_1}00\ldots01|}_{i_2}\;\cdots\cdots\;0\ldots0\,1|}_{i_k}00\ldots00$$

Elements of vector space V map to n-dimensional subspace of the algebra: $\boldsymbol{v} = \sum_{i=1}^{n} v_i\boldsymbol{e}_i$. Elements with unit norm $\|\boldsymbol{v}\| = 1$ [2] have inverse, (for $\mathbb{Cl}(n)$, $\boldsymbol{v}^{-1} = -\boldsymbol{v}$ because $-\boldsymbol{v}\cdot\boldsymbol{v} = \mathbf{1}$) and product of even number of such elements is some group. It is called *spinor group Spin*(n) and it has 2-1 isomorphism with group of n-dimensional rotations, $SO(n)$.

In the definition of spinor group is used only multiplication of *even* number of elements of algebra $\mathbb{Cl}(n)$. It is possible to build 2^{n-1} different compositions eq.(2) with even number of generators of $\mathbb{Cl}(n)$ and all such compositions produce *even subalgebra* $\mathbb{Cl}^e(n)$ of Clifford algebra and really spinor group is subspace of the smaller algebra. But it can be shown that $\mathbb{Cl}^e(n)$

[1] Let the product is $\mathbf{1}$ for $k = 0$

[2] More generally $|Q(\boldsymbol{v})| = 1$

is isomorphic with $\mathbb{Cl}(n-1)$ and the algebra also may be used for representation of $Spin(n)$ group. The property is mentioned here to explain why it is important sometime to consider Clifford algebras with dimension reduced by unit, for example $\mathbb{Cl}(2)$ is enough to build spinor group for 3D rotations.

It is possible also to introduce yet another algebra, $\mathbb{Cl}_+(n)$, if to change sign of square of all $\boldsymbol{e}_k$ to $+\mathbf{1}$:

$$\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i = 2\delta_{ij}; \implies \boldsymbol{e}_i\boldsymbol{e}_j = -\boldsymbol{e}_j\boldsymbol{e}_i \ (i \neq j), \quad \underline{\boldsymbol{e}_k^2 = \mathbf{1}} \tag{3}$$

The two real algebras are not equivalent.

Clifford algebra $\mathbb{Cl}(m,l)$ corresponds to more general, pseudo-Euclidean case (${\boldsymbol{e}_k}^2 = \pm\mathbf{1}$):

$$\boldsymbol{e}_i\boldsymbol{e}_j = -\boldsymbol{e}_j\boldsymbol{e}_i \ (i \neq j), \quad \boldsymbol{e}_k^2 = -\mathbf{1} \ (k \leq m), \quad \boldsymbol{e}_k^2 = \mathbf{1} \ (k > m) \tag{4}$$

Two previous examples are special cases of the definition: $\mathbb{Cl}(n) = \mathbb{Cl}(n,0)$, $\mathbb{Cl}_+(n) = \mathbb{Cl}(0,n)$.

It is possible also to consider 'degenerated' case ($Q(x) \equiv 0$) :

$$\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i = 0 \quad \forall i,j \ (\boldsymbol{e}_k^2 = 0) \tag{5}$$

It is known also as *Grassmann algebra* Λ^n (with notation $a \wedge b$ is used instead of $a\,b$, $\boldsymbol{e}_i \wedge \boldsymbol{e}_j = -\boldsymbol{e}_j \wedge \boldsymbol{e}_i$) and usually it is considered separately. One of applications of Grassmann algebras is related with description of subspaces of n-dimensional vector space.

It was real Clifford algebras and the complexification of any algebra $\mathbb{Cl}(l, n-l)$ produces the same universal Clifford algebra $\mathbb{Cl}(n,\mathbb{C})$ because it is always possible to 'correct' sign of any $\boldsymbol{e}_k^2$ by substitution $\tilde{\boldsymbol{e}}_k = i\boldsymbol{e}_k$, $\tilde{\boldsymbol{e}}_k^2 = -\boldsymbol{e}_k^2$.

Let us use notation $\boldsymbol{e}_k$ for elements of basis with $\boldsymbol{e}_k^2 = -\mathbf{1}$ and $\overset{+}{\boldsymbol{e}}_k$ for $\overset{+}{\boldsymbol{e}}{}_k^2 = \mathbf{1}$, $\overset{+}{\boldsymbol{e}}_k = i\boldsymbol{e}_k$.

—

The following constructions of Grassmann algebras are used further: Let us consider even-dimensional universal Clifford algebra $\mathbb{Cl}(2n,\mathbb{C})$. It is possible to build a Grassmann algebra Λ^n with using n-generators:

$$\boldsymbol{d}_l = \boldsymbol{e}_{2l} + i\boldsymbol{e}_{2l+1}; \implies \boldsymbol{d}_k\boldsymbol{d}_j = -\boldsymbol{d}_j\boldsymbol{d}_k \quad \forall k,j \quad (\boldsymbol{d}_k^2 = 0) \tag{6}$$

It is possible also to define two different algebras: Λ^n_+ with generators a_l and Λ^n_- with $a^\dagger_l$:

$$\boldsymbol{a}_l = \frac{\overset{+}{\boldsymbol{e}}_{2l} + i\overset{+}{\boldsymbol{e}}_{2l+1}}{2}, \quad \boldsymbol{a}^\dagger_l = \frac{\overset{+}{\boldsymbol{e}}_{2l} - i\overset{+}{\boldsymbol{e}}_{2l+1}}{2} \tag{7}$$

with properties:

$$\boxed{\{\boldsymbol{a}_i, \boldsymbol{a}_j\} = 0, \quad \{\boldsymbol{a}^\dagger_i, \boldsymbol{a}^\dagger_j\} = 0, \quad \{\boldsymbol{a}^\dagger_i, \boldsymbol{a}_j\} = \delta_{ij}} \tag{8}$$

where $\{a, b\} \equiv (a\,b + b\,a)$. A nontrivial case is $\{\boldsymbol{a}^\dagger_l, \boldsymbol{a}_l\} = \mathbf{1}$ also can be simply checked:

$$\begin{aligned}
4\,\boldsymbol{a}_l\,\boldsymbol{a}^\dagger_l &= (\overset{+}{\boldsymbol{e}}_{2l} + i\overset{+}{\boldsymbol{e}}_{2l+1})(\overset{+}{\boldsymbol{e}}_{2l} - i\overset{+}{\boldsymbol{e}}_{2l+1}) = \\
&= \overset{+}{\boldsymbol{e}}{}^2_{2l} + i\overset{+}{\boldsymbol{e}}_{2l+1}\overset{+}{\boldsymbol{e}}_{2l} - i\overset{+}{\boldsymbol{e}}_{2l}\overset{+}{\boldsymbol{e}}_{2l+1} + \overset{+}{\boldsymbol{e}}{}^2_{2l+1} = 2 - 2i\,\overset{+}{\boldsymbol{e}}_{2l}\overset{+}{\boldsymbol{e}}_{2l+1} \\
4\,\boldsymbol{a}^\dagger_l\,\boldsymbol{a}_l &= (\overset{+}{\boldsymbol{e}}_{2l} - i\overset{+}{\boldsymbol{e}}_{2l+1})(\overset{+}{\boldsymbol{e}}_{2l} + i\overset{+}{\boldsymbol{e}}_{2l+1}) = \\
&= \overset{+}{\boldsymbol{e}}{}^2_{2l} - i\overset{+}{\boldsymbol{e}}_{2l+1}\overset{+}{\boldsymbol{e}}_{2l} + i\overset{+}{\boldsymbol{e}}_{2l}\overset{+}{\boldsymbol{e}}_{2l+1} + \overset{+}{\boldsymbol{e}}{}^2_{2l+1} = 2 + 2i\,\overset{+}{\boldsymbol{e}}_{2l}\overset{+}{\boldsymbol{e}}_{2l+1}
\end{aligned}$$

1.2 Two-dimensional case.

Let us consider real Clifford algebras first. The 4D real algebra $\mathbb{Cl}(2)$ is described by elements $\mathfrak{i} = \boldsymbol{e}_1$, $\mathfrak{j} = \boldsymbol{e}_2$, $\mathfrak{k} = \boldsymbol{e}_1\boldsymbol{e}_2$ and unit element $\mathbf{1}$. The elements satisfy relations: $\mathfrak{i}^2 \overset{\text{def}}{=} -\mathbf{1}$, $\mathfrak{j}^2 \overset{\text{def}}{=} -\mathbf{1}$, $-\mathfrak{j}\mathfrak{i} \overset{\text{def}}{=} \mathfrak{i}\mathfrak{j} \overset{\text{def}}{=} \mathfrak{k}$, $\mathfrak{k}^2 = \mathfrak{i}\mathfrak{j}\mathfrak{k} = -\mathbf{1}$, $\mathfrak{i}\mathfrak{k} = \mathfrak{i}\mathfrak{i}\mathfrak{j} = -\mathfrak{j} = \mathfrak{j}\mathfrak{i}\mathfrak{i} = -\mathfrak{k}\mathfrak{i}$, $\mathfrak{j}\mathfrak{k} = -\mathfrak{j}\mathfrak{j}\mathfrak{i} = \mathfrak{i} = -\mathfrak{i}\mathfrak{j}\mathfrak{j} = -\mathfrak{k}\mathfrak{j}$ and the equations define well known 4D *quaternions algebra* $\mathbb{H}$ introduced by Hamilton at 1843 as a generalization of complex numbers.

Another 4D real Clifford algebra $\mathbb{Cl}_+(2)$ is generated by elements $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} = \boldsymbol{u}\boldsymbol{v}$ with defining relations are: $\boldsymbol{u}^2 \overset{\text{def}}{=} \mathbf{1}$, $\boldsymbol{v}^2 \overset{\text{def}}{=} \mathbf{1}$, $\boldsymbol{u}\boldsymbol{v} \overset{\text{def}}{=} -\boldsymbol{v}\boldsymbol{u} \overset{\text{def}}{=} \boldsymbol{w}$ and with implications: $\boldsymbol{w}^2 = \boldsymbol{u}\boldsymbol{v}\boldsymbol{w} = -\mathbf{1}$, $\boldsymbol{v}\boldsymbol{w} = \boldsymbol{v}\boldsymbol{v}\boldsymbol{u} = \boldsymbol{u} = \boldsymbol{u}\boldsymbol{v}\boldsymbol{v} = -\boldsymbol{w}\boldsymbol{v}$, $\boldsymbol{u}\boldsymbol{w} = -\boldsymbol{u}\boldsymbol{u}\boldsymbol{v} = -\boldsymbol{v} = -\boldsymbol{v}\boldsymbol{u}\boldsymbol{u} = -\boldsymbol{w}\boldsymbol{u}$.

It is possible to consider $\boldsymbol{v}$ and $\boldsymbol{w}$ as generators of some new Clifford algebra and because $\boldsymbol{w}^2 = -\mathbf{1}$ it is $\mathbb{Cl}(1,1)$ and so the algebra is isomorphic with $\mathbb{Cl}_+(2) \equiv \mathbb{Cl}(0,2)$.

The $\mathbb{Cl}_+(2)$ is isomorphic with 4D algebra $\mathbb{R}(2{\times}2)$ of all 2×2 real matrices. It is enough to choose: $\boldsymbol{u} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\boldsymbol{v} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\boldsymbol{w} = \boldsymbol{u}\boldsymbol{v} = -\boldsymbol{v}\boldsymbol{u} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\mathbf{1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is unit matrix.

The complexification of the algebras, $\mathbb{Cl}(2,\mathbb{C})$ is isomorphic with algebra $\mathbb{C}(2{\times}2)$ of all complex 2×2 matrices and representation by Pauli matrices is: $\overset{+}{\boldsymbol{e}}_1 = \boldsymbol{\sigma}_x$, $\overset{+}{\boldsymbol{e}}_2 = \boldsymbol{\sigma}_y$, $\overset{+}{\boldsymbol{e}}_1\overset{+}{\boldsymbol{e}}_2 = i\boldsymbol{\sigma}_z$,

$$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

with standard relations $\boldsymbol{\sigma}_x^2 = \boldsymbol{\sigma}_y^2 = \boldsymbol{\sigma}_z^2 = \mathbf{1}$, $\boldsymbol{\sigma}_x\boldsymbol{\sigma}_y = i\boldsymbol{\sigma}_z = -\boldsymbol{\sigma}_y\boldsymbol{\sigma}_x$, $\boldsymbol{\sigma}_z\boldsymbol{\sigma}_x = i\boldsymbol{\sigma}_y = -\boldsymbol{\sigma}_x\boldsymbol{\sigma}_z$, $\boldsymbol{\sigma}_y\boldsymbol{\sigma}_z = i\boldsymbol{\sigma}_x = -\boldsymbol{\sigma}_z\boldsymbol{\sigma}_y$.

The real Clifford algebras $\mathbb{Cl}(2) \equiv \mathbb{Cl}(2,0) \cong \mathbb{H}$ and $\mathbb{Cl}_+(2) \equiv \mathbb{Cl}(0,2) \cong \mathbb{Cl}(1,1) \cong \mathbb{R}(2{\times}2)$ are subalgebras of $\mathbb{Cl}(2,\mathbb{C}) \cong \mathbb{C}(2{\times}2)$:

$$\mathbb{Cl}_+(2) \hookrightarrow \mathbb{Cl}(2,\mathbb{C}): \quad \boldsymbol{u} \to \boldsymbol{\sigma}_x,\ \boldsymbol{v} \to \boldsymbol{\sigma}_z,\ \boldsymbol{w} \to \boldsymbol{\sigma}_y/i \tag{10}$$
$$\mathbb{Cl}(2) \hookrightarrow \mathbb{Cl}(2,\mathbb{C}): \quad \mathfrak{i} \to i\boldsymbol{\sigma}_x,\ \mathfrak{j} \to i\boldsymbol{\sigma}_y,\ \mathfrak{k} \to i\boldsymbol{\sigma}_z \tag{11}$$

Really, the eq.(11) corresponds to usual representation of quaternions by complex 2×2 matrices: $\mathfrak{i} = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$, $\mathfrak{j} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $\mathfrak{k} = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, but it is useful sometime to do not use some particular matrix notation and consider quaternions as an *abstract* linear algebra defined by relation with $\mathfrak{i}$, $\mathfrak{j}$ and $\mathfrak{k}$ introduced earlier:

$$\mathfrak{i}^2 = \mathfrak{j}^2 = -\mathbf{1},\ \mathfrak{i}\cdot\mathfrak{j} = -\mathfrak{j}\cdot\mathfrak{i} = \mathfrak{k};\ q_0\mathbf{1} + q_1\mathfrak{i} + q_2\mathfrak{j} + q_3\mathfrak{k} \in \mathbb{H};\ (q_k \in \mathbb{R}) \tag{12}$$

For example, such definition makes more clear difference between 4D *real* algebra of quaternions and 4D complex Pauli algebra.

It should be mentioned, that the Pauli algebra can be considered also as 8D *real* Clifford algebra $\mathbb{Cl}_+(3)$ with relations: $\boldsymbol{\sigma}_x^2 = \boldsymbol{\sigma}_y^2 = \boldsymbol{\sigma}_z^2 = \mathbf{1}$. An element $\mathbf{i} \equiv \boldsymbol{\sigma}_x\boldsymbol{\sigma}_y\boldsymbol{\sigma}_z$ of the algebra commutes with any other element and corresponds to $i\mathbf{1}$ of complex algebra $\mathbb{Cl}(2,\mathbb{C})$.

1.3 Constriction of $\mathbb{Cl}(2n,\mathbb{C})$.

Let us recall that for n-dimensional linear space V with basis $e_1,\ldots,e_n$ and m-dimensional linear space W with basis $e'_1,\ldots,e'_m$ the *tensor product* $V \otimes W$ of the spaces can be introduced as $m{\cdot}n$-dimensional space with basis: $e_i \otimes e'_j$.

Similarly, for n-dimensional linear space V with basis $e_1,\ldots,e_n$ the *tensor power* $V^{\otimes k}$ of k copies of such space can be introduced as n^k-dimensional space with basis: $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$, $1 \le i_l \le n$.

If the linear spaces have structure of some algebras A and B it is possible to build new nm-dimensional algebra $A \otimes B$ with composition is defined as: $(a \otimes b) \cdot (c \otimes d) \equiv (ac) \otimes (bd)$ for elements of basis and extended to all elements by distributivity. The n^k dimensional tensor power $A^{\otimes k}$ is defined analogously.

Let us show direct construction of $\mathbb{C}\mathrm{l}(2n, \mathbb{C})$ as algebra $\mathbb{C}(2^n{\times}2^n)$ of all $2^n \times 2^n$ matrices. It can be done by two steps with using Pauli matrices. First step is isomorphism $\mathbb{C}\mathrm{l}(2n, \mathbb{C}) \cong \mathbb{C}(2{\times}2)^{\otimes n}$ and second is $\mathbb{C}(2{\times}2)^{\otimes n} \cong \mathbb{C}(2^n{\times}2^n)$.

It is possible to show first isomorphism by direct construction of $2n$ generators of $\mathbb{C}\mathrm{l}(2n, \mathbb{C})$:

$$\begin{aligned} \overset{+}{\boldsymbol{e}}_{2k} &= \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{n-k-1} \otimes \boldsymbol{\sigma}_x \otimes \underbrace{\boldsymbol{\sigma}_z \otimes \cdots \otimes \boldsymbol{\sigma}_z}_{k} \\ \overset{+}{\boldsymbol{e}}_{2k+1} &= \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{n-k-1} \otimes \boldsymbol{\sigma}_y \otimes \underbrace{\boldsymbol{\sigma}_z \otimes \cdots \otimes \boldsymbol{\sigma}_z}_{k} \end{aligned} \tag{13}$$

To check that all the $2n$ generators are anticommutative it is possible to introduce the construction inductively. For $n = 1$ the $\mathbb{C}\mathrm{l}(2, \mathbb{C})$ has two generators $\boldsymbol{\sigma}_x$ and $\boldsymbol{\sigma}_y$. If the suggestion is shown for some $l \geq 1$ and we have $2l$ generators $\overset{+}{\boldsymbol{e}}_1, \ldots, \overset{+}{\boldsymbol{e}}_{2l} \in \mathbb{C}(2{\times}2)^{\otimes l} \cong \mathbb{C}\mathrm{l}(2l, \mathbb{C})$ then for $l + 1$ it is also possible to choose $2l + 2$ generators in $\mathbb{C}(2{\times}2)^{\otimes l+1}$: *i.e.* $2l$ anticommutative elements $\overset{+}{\boldsymbol{e}}_1 \otimes \boldsymbol{\sigma}_z, \ldots, \overset{+}{\boldsymbol{e}}_{2l} \otimes \boldsymbol{\sigma}_z$ together with two elements $\mathbf{1}^{\otimes l} \otimes \boldsymbol{\sigma}_x$ and $\mathbf{1}^{\otimes l} \otimes \boldsymbol{\sigma}_y$, where $\mathbf{1}^{\otimes l} = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{l}$ is unit of $\mathbb{C}\mathrm{l}(2l, \mathbb{C})$. So $\mathbb{C}\mathrm{l}(2l + 2, \mathbb{C}) \cong \mathbb{C}(2{\times}2)^{\otimes l+1}$.

The isomorphism of $\mathbb{C}(2{\times}2)^{\otimes n} \cong \mathbb{C}(2^n{\times}2^n)$ can be also shown by induction. It is true for $n = 1$ and if it is proven for some $l \geq 1$ then any $A \in \mathbb{C}(2^l{\times}2^l)$ is represented by some $2^l \times 2^l$ matrix $\mathbf{A}$. It is possible to check that $\mathbb{C}(2^{l+1}{\times}2^{l+1}) \cong \mathbb{C}(2{\times}2) \otimes \mathbb{C}(2^l{\times}2^l)$, if to define operation:

$$\begin{aligned} \mathbf{a} &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{C}(2{\times}2), \qquad \mathbf{A} \in \mathbb{C}(2^l{\times}2^l), \\ \mathbf{a} \overset{\text{mat}}{\otimes} \mathbf{A} &\equiv \begin{pmatrix} (a_{11}\mathbf{A}) & (a_{12}\mathbf{A}) \\ (a_{21}\mathbf{A}) & (a_{22}\mathbf{A}) \end{pmatrix} \in \mathbb{C}(2^{l+1}{\times}2^{l+1}) \end{aligned} \tag{14}$$

It can be checked directly, $(\mathbf{a} \overset{\text{mat}}{\otimes} \mathbf{A}) \cdot (\mathbf{b} \overset{\text{mat}}{\otimes} \mathbf{B}) = \mathbf{ab} \overset{\text{mat}}{\otimes} \mathbf{AB}$ if $\mathbf{a}, \mathbf{b}$ are basis elements: $\mathbf{1}, \boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z$ and it is extended for all elements by distributivity.

2 Quantum gates.

Any quantum n-gate is described by some unitary $2^n \times 2^n$ matrix. It acts on n-qubits in 2^n-dimensional complex vector space.

Matrices of all $2^{2n} = 4^n$ possible compositions of $2n$ generators in eq.(13) are unitary. Really, any such element has form:

$$\boldsymbol{e}_{l_1}\boldsymbol{e}_{l_2}\cdots\boldsymbol{e}_{l_k} \xrightarrow{0\leqslant k\leqslant 2n} \boldsymbol{e}_{\boldsymbol{I}} \equiv \boldsymbol{\sigma}_{i_1}\otimes\boldsymbol{\sigma}_{i_2}\otimes\cdots\otimes\boldsymbol{\sigma}_{i_n} \tag{15}$$

where $\boldsymbol{\sigma}_{i_k}$ is one of four basis elements: $\mathbf{1}, \boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z$. In matrix representation used here $\boldsymbol{\sigma}_i^* = \boldsymbol{\sigma}_i$, $\boldsymbol{\sigma}_i^2 = \mathbf{1}$. For any $\boldsymbol{\sigma}_i$ and matrix $\mathbf{A}^* = \mathbf{A}$ the eq.(14) implies $(\boldsymbol{\sigma}_i \overset{\text{mat}}{\otimes} \mathbf{A})^* = (\boldsymbol{\sigma}_i \overset{\text{mat}}{\otimes} \mathbf{A})$ and so $\boldsymbol{e}_{\boldsymbol{I}}^* = \boldsymbol{e}_{\boldsymbol{I}}$. Because also $\boldsymbol{e}_{\boldsymbol{I}}^2 = \mathbf{1}$, all the matrices for 4^n elements satisfy equations: $\boldsymbol{e}_{\boldsymbol{I}}^* = \boldsymbol{e}_{\boldsymbol{I}} = \boldsymbol{e}_{\boldsymbol{I}}^{-1}$, they are not only Hermitian ($\mathbf{A}^* = \mathbf{A}$), but unitary too ($\mathbf{A}^* = \mathbf{A}^{-1}$).

Such construction of unitary matrices may be not very general, because it produces finite number of gates, but it is possible to consider some analogous of 'Hamiltonian approach', if to use Hermitian matrices $\mathbf{H}$ and then $\mathbf{U} = \exp(i\,\mathbf{H})$ is unitary. It is useful because whole 4^n dimensional space of Hermitian matrices is produced by combinations up to 4^n elements $\boldsymbol{e}_{\boldsymbol{I}}$ described by eq.(15) with real coefficients:

$$\mathcal{H} = \Big\{\mathbf{H} \;:\; \mathbf{H} = \sum\nolimits_{\boldsymbol{I}} c_{\boldsymbol{I}}\boldsymbol{e}_{\boldsymbol{I}}; \quad c_{\boldsymbol{I}} \in \mathbb{R}\Big\} \tag{16}$$

$\mathbf{H} \in \mathcal{H}$, $\dim\mathcal{H} = 4^n$.

It is necessary also to define action of an element of $\mathbb{Cl}(2n,\mathbb{C}) \cong \mathbb{C}(2^n \times 2^n)$ on n-qubit register as an action of $2^n \times 2^n$ matrix on complex vector in 2^n-dimensional complex space. Let us define it for basis of 2^n vector space

$$e_L = e_{l_1} \otimes e_{l_2} \otimes \cdots \otimes e_{l_n}, \quad l_k = 0, 1 \tag{17}$$

or with notation used in quantum information science

$$|L\rangle = |l_1\, l_2 \ldots l_n\rangle = |l_1\rangle \otimes |l_2\rangle \otimes \cdots \otimes |l_n\rangle, \quad l_k = 0, 1 \tag{18}$$

$$e_0 = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_1 = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{19}$$

and 4^n-dimensional basis eq.(15) of Clifford algebra:

$$\boldsymbol{g}_I = \boldsymbol{\sigma}_{i_1} \otimes \boldsymbol{\sigma}_{i_2} \otimes \cdots \otimes \boldsymbol{\sigma}_{i_n}, \quad \boldsymbol{\sigma}_{i_k} \in \mathbf{1}, \boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z \tag{20}$$

The action is defined for the elements of basis component-wise:

$$\boldsymbol{g}_I|L\rangle = \big(\boldsymbol{\sigma}_{i_1}|l_1\rangle\big) \otimes \big(\boldsymbol{\sigma}_{i_2}|i_2\rangle\big) \otimes \cdots \otimes \big(\boldsymbol{\sigma}_{i_n}|l_n\rangle\big) \tag{21}$$

where $\boldsymbol{\sigma}_{i_k}|l_k\rangle$ is action of 2×2 complex matrix $\boldsymbol{\sigma}$ on 2D complex vector $|0\rangle$ or $|1\rangle$. The action of arbitrary element (gate) to a general vector (*entangled* state) is produced from eq.(21) by linearity:

$$\boldsymbol{G}|\psi\rangle = \sum_I a_I \boldsymbol{g}_I \sum_L c_L|L\rangle = \sum_{I,L} (a_I c_L)\, \boldsymbol{g}_I|L\rangle \tag{22}$$

So, any linear transformation of 2^n-dimensional space can be represented by elements of Clifford algebra with $2n$ generators

$$\mathbb{C}\mathbb{l}(2n, \mathbb{C}) \cong \mathbb{C}(2^n{\times}2^n) : \mathbb{C}^{2^n} \to \mathbb{C}^{2^n} \tag{23}$$

To build particular matrix for some quantum n-gate it is possible to start with simple 2×2 $\boldsymbol{\sigma}$-matrices for $\mathbb{C}\mathbb{l}(2,\mathbb{C})$ and qubit in basis eq.(19). After it, recursively $1, \ldots, n$ on each step it is considered matrices and vectors with doubled size:

$$(\Box)\cdot|...\rangle \xrightarrow{k\to k+1} \begin{pmatrix} (\Box) & (\Box) \\ (\Box) & (\Box) \end{pmatrix} \cdot \left| \begin{matrix} (|0...\rangle) \\ (|1...\rangle) \end{matrix} \right\rangle \tag{24}$$

For example of application the theory of Clifford algebras to quantum information science let us use now other basis with dual Grassmann elements introduced by eq.(7,8) to describe approach with occupation numbers used by Feynman [3, 4].

Let us apply eq.(7) to expressions for Clifford generators eq.(13):

$$\begin{aligned} \boldsymbol{a}_k &= \mathbf{1}^{\otimes n-k-1} \otimes \frac{\boldsymbol{\sigma}_x + i\boldsymbol{\sigma}_y}{2} \otimes \boldsymbol{\sigma}_z^{\otimes k} \\ \boldsymbol{a}_k^\dagger &= \mathbf{1}^{\otimes n-k-1} \otimes \frac{\boldsymbol{\sigma}_x - i\boldsymbol{\sigma}_y}{2} \otimes \boldsymbol{\sigma}_z^{\otimes k} \end{aligned} \tag{25}$$

The matrices $\boldsymbol{a} = \frac{\boldsymbol{\sigma}_x+i\boldsymbol{\sigma}_y}{2} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\boldsymbol{a}^\dagger = \frac{\boldsymbol{\sigma}_x-i\boldsymbol{\sigma}_y}{2} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ correspond to *annihilation* and *creation* operators used by Feynman for description of quantum bit and commutation relations in eq.(8) correspond to correct extension of the operators for n fermions.

The approach is analogue of *second quantization* and some advantages of the paradigm are discussed already in [6].

On the other hand, due to isomorphism $\mathbb{C}\mathbf{l}(2n,\mathbb{C}) \cong \mathbb{C}(2^n{\times}2^n)$ described above, the same elements produce basis for description of quantum gates by 2×2 complex matrices used in S-matrix approach originated by Deutsch [5] *et al.*

The operator $\boldsymbol{a}$ of annihilation and $\boldsymbol{a}^\dagger$ of creation can be also written as (here $\mathbf{0}$ is zero vector):

$$\boldsymbol{a} = |0\rangle\langle 1|; \qquad \boldsymbol{a}^\dagger = |1\rangle\langle 0| \tag{26}$$

$$\boldsymbol{a}|1\rangle = |0\rangle, \quad \boldsymbol{a}|0\rangle = \mathbf{0}; \qquad \boldsymbol{a}^\dagger|0\rangle = |1\rangle, \quad \boldsymbol{a}^\dagger|1\rangle = \mathbf{0} \tag{27}$$

3 Some remarks

The construction of tensor product of algebras depends on *field of scalars* (say $\mathbb{R}$ or $\mathbb{C}$) used in definition of the algebras. And if algebras could be considered either as complex or as real, then it is possible to use two kinds of tensor products:

$$A \otimes_{\mathbb{C}} B \not\cong A \otimes_{\mathbb{R}} B \tag{28}$$

For example, for algebra of Pauli matrices considered as 8D real algebra the n-th tensor power is $8^n = 2^{3n}$D real algebra and it is not equivalent to construction used in the paper with 2^{2n} complex or 2^{2n+1} real dimension. The fact clarify difference between spaces used in the paper and real tensor product in [7] there real tensor product was used.

With using of real tensor product it is also possible to express space of quantum n-gates and $\mathbb{C}\mathbf{l}(2n,\mathbb{C})$:

$$\boldsymbol{G}_n \in \mathbb{C} \otimes \mathbb{R}(2{\times}2)^{\otimes n} \tag{29}$$

or:

$$\boldsymbol{G}_n \in \mathbb{C} \otimes \mathbb{H}^{\otimes n} \tag{30}$$

where $\mathbb{C}$ is considered as 2D real and $\mathbb{H}$ as 4D real algebras.

Acknowledgements

Author is grateful to organizers of *TMR99 Network School on "Quantum Computation and Quantum Information Theory"* for invitation, financial support and hospitality.

References

[1] A. R. Calderbank, E. M. Rains, P. W. Shor, N. J. A. Sloane, Quantum Error Correction and Orthogonal Geometry, *Phys. Rev. Lett.* **78** (1997), 405–408.

[2] J. E. Gilbert, M. A. M. Murray, *Clifford algebras and Dirac operators in harmonic analysis*, Cambridge University Press, 1991.

[3] Richard Feynman, Simulating Physics with Computers, *Internal Journal of Theoretical Physics* **21** (1982), 467–488.

[4] Richard Feynman, Quantum-Mechanical Computers, *Foundations of Physics* **16** (1986), 507–531.

[5] David Deutsch, Quantum computational networks, *Proceedings of Royal Society of London* **A 425**, (1989), 73–90.

[6] A. Yu. Vlasov, Quantum theory of computation and relativistic physics, *Proceedings of the Fourth Workshop on Physics and Computation* (1996), 332–333 [quant-ph/9701027 – full version]

[7] S. S. Somaroo, D. G. Cory, T. F. Havel, Expressing the operations of quantum computing in multiparticle geometric algebra, *Phys. Lett.* **A 240** (1998), 1–7.